Lecture No. 11

The Finite Element Method

The FEM involves the following general steps

1) Develop a weighted residual formulation (also involved selection of type of weighting).

2) Select type of interpolation (order can be arbitrary in the implementation of the code).

3) Discretization of the problem by selection of elements interconnected at certain nodal points.

4) Evaluation of the matrices of the elements.

5) Formulation of the complete matrix globally.

6) Application of b.c.'s

7) Solution of the resulting system of equations

8) Calculation of any other functions based on nodal unknowns.

Example Finite Element Application

- The general elemental approach allows the use of any order interpolation. It is a more formal and general approach of viewing the matrix set up procedures.
- Again consider the example problem

$$\frac{dT}{dt} + 2T - 1 = 0$$

- The weighted residual formulation assuming Galerkin is:

$$\int_{\Omega_t} \left(\frac{d\hat{T}}{dt} + 2\hat{T} - 1 \right) \delta\hat{T} dt = 0$$

- We will assume that for each element *n*, Lagrange interpolation of order $M - 1$ is used. For element (*n*):

$$T \cong \hat{T}^{(n)} = T_1^{(n)}\phi_1^{(n)} + T_2^{(n)}\phi_1^{(n)} + \cdots + T_M^{(n)}\phi_M^{(n)} = \sum_{i=1}^{M} T_i^{(n)}\phi_i^{(n)}$$

- Within each element *n* this may be written in <u>vector</u> form as:

$$\hat{T}^{(n)} = \underline{\phi}^{(n)} \underline{T}^{(n)}$$

$$\Rightarrow$$

$$\hat{T}^{(n)} = \left[\phi_1^{(n)} \phi_2^{(n)} \phi_3^{(n)} \ldots \phi_M^{(n)} \right] \begin{bmatrix} T_1^{(n)} \\ T_2^{(n)} \\ \cdot \\ T_M^{(n)} \end{bmatrix}$$

$\hat{T}^{(n)}$ is a scalar

$\underline{\phi}^{(n)}$ represents the vector of elemental interpolating functions

$\underline{T}^{(n)}$represents the vector of elemental unknowns (which equal the variable at each node)

$\underline{\phi}^{(n)}$ can represent any order Lagrange interpolation

The order selected determines the size of $\underline{\phi}^{(n)}$ and $\underline{T}^{(n)}$ and the components $\phi_i^{(n)}$

- Since each set of interpolating functions $\underline{\phi}^{(n)}$ is identically equal to zero outside element *n*, over the entire domain we have:

$$\hat{T} = \sum_{n=1}^{\#\,elements} \hat{T}^{(n)} = \sum_{n=1}^{\#\,elements} \underline{\phi}^{(n)} \underline{T}^{(n)}$$

- Similarly we have:

$$\delta\hat{T}^{(n)} = \underline{\phi}^{(n)} \delta\underline{T}^{(n)}$$

where $\delta\underline{T}^{(n)}$ can represent any vector of arbitrary coefficients.

$$\delta\hat{T}^{(n)} = \sum_{n=1}^{\#\,elements} \delta\hat{T}^{(n)} = \sum_{n=1}^{\#\,elements} \underline{\phi}^{(n)} \delta\underline{T}^{(n)}$$

- Recall that for the Galerkin weighted residual method we expressed:

$$\delta u = \delta\alpha_1 \phi_1 + \delta\alpha_2 \phi_2 + \cdots$$

and that

$$\langle \varepsilon_I, \delta u \rangle = 0$$

had to be true for all values of $\delta\alpha_i$. Therefore vector $\delta\underline{\hat{T}}$ is arbitrary.

- Thus:

$$\int_{\Omega_t} \left\{ \frac{d}{dt}\left(\sum_{n=1}^{\#el} \underline{\phi}^{(n)} \underline{T}^{(n)} \right) + 2\left(\sum_{n=1}^{\#el} \underline{\phi}^{(n)} \underline{T}^{(n)} \right) - 1 \right\} \left(\sum_{n=1}^{\#el} \underline{\phi}^{(n)} \delta \underline{T}^{(n)} \right) dt = 0$$

Hence the coefficients $\underline{T}^{(n)}$ are independent of the variable *t* and are constants. Therefore the variation in time is described by $\underline{\phi}^{(n)}$ which are functions of dependent variable *t*.

Therefore:

$$\frac{d}{dt}\left(\sum_{n=1}^{\#el} \underline{\phi}^{(n)} \underline{T}^{(n)} \right) = \sum_{n=1}^{\#el} \underline{\phi}_{,t}^{(n)} \underline{T}^{(n)}$$

Hence:

$$\int_{\Omega_t} \left\{ \left(\sum_{n=1}^{\#el} \underline{\phi}_{,t}^{(n)} \underline{T}^{(n)} \right) \left(\sum_{n=1}^{\#el} \underline{\phi}^{(n)} \delta \underline{T}^{(n)} \right) + 2\left(\sum_{n=1}^{\#el} \underline{\phi}^{(n)} \underline{T}^{(n)} \right) \left(\sum_{n=1}^{\#el} \underline{\phi}^{(n)} \delta \underline{T}^{(n)} \right) - \left(\sum_{n=1}^{\#el} \underline{\phi}^{(n)} \delta \underline{T}^{(n)} \right) \right\} dt = 0$$

- Since the interpolating functions are defined as nonzero only over the element *n*, we note that:

$$\left(\sum_{n=1}^{\#el} \underline{\phi}_{,t}^{(n)} \underline{T}^{(n)}\right)\left(\sum_{n=1}^{\#el} \underline{\phi}^{(n)} \delta \underline{T}^{(n)}\right) = \sum_{n=1}^{\#el} \left(\underline{\phi}_{,t}^{(n)} \underline{T}^{(n)} \underline{\phi}^{(n)} \delta \underline{T}^{(n)}\right)$$

- Furthermore when we integrate we need integrate only over each element and then sum over all elements. Therefore

$$\sum_{n=1}^{\#el} \left\{ \int_{\Omega_{el}^{(n)}} \left(\underline{\phi}_{,t}^{(n)} \underline{T}^{(n)} \underline{\phi}^{(n)} \delta \underline{T}^{(n)} + 2\underline{\phi}^{(n)} \underline{T}^{(n)}\, \underline{\phi}^{(n)} \delta\, \underline{T}^{(n)} - \underline{\phi}^{(n)} \delta \underline{T}^{(n)}\right) dt \right\} = 0$$

- Usually we go directly to this previous step from the weighted residual formulation.

- We can re-arrange things somewhat further:

$$\underline{\phi}_{,t}^{(n)}\underline{T}^{(n)}\underline{\phi}^{(n)}\delta\underline{T}^{(n)} = \underline{\phi}^{(n)}\delta\underline{T}^{(n)}\underline{\phi}_{,t}^{(n)}\underline{T}^{(n)}$$

$$= \left[\phi_1^{(n)}\phi_2^{(n)}\cdots\right]\begin{bmatrix}\delta T_1^{(n)}\\ \delta T_2^{(n)}\\ \vdots\\ .\end{bmatrix}\left[\phi_{1,t}^{(n)}\quad\phi_{1,t}^{(n)}\cdots\right]\begin{bmatrix}T_1^{(n)}\\ T_2^{(n)}\\ \vdots\\ .\end{bmatrix}$$

$$= \delta\underline{T}^{(n)^T}\underline{\phi}^{(n)^T}\underline{\phi}_{,t}^{(n)}\underline{T}^{(n)}$$

- Substituting we obtain:

$$\sum_{n=1}^{\#el}\left\{\int_{\Omega_{el}^{(n)}}\left[\delta\underline{T}^{(n)^T}\underline{\phi}^{(n)^T}\underline{\phi}_{,t}^{(n)}\underline{T}^{(n)} + \delta\underline{T}^{(n)^T}\underline{\phi}^{(n)^T}2\underline{\phi}^{(n)}\underline{T}^{(n)} - \delta\underline{T}^{(n)^T}\underline{\phi}^{(n)^T}\right]dt\right\} = 0$$

Factoring out $\delta\underline{T}^{(n)^T}$ and rewrite:

$$\sum_{n=1}^{\#el}\delta\underline{T}^{(n)^T}\left\{\left[\int_{\Omega_{el}^{(n)}}\underline{\phi}^{(n)^T}\underline{\phi}_{,t}^{(n)}\,dt\right]\underline{T}^{(n)} + \left[2\int_{\Omega_{el}^{(n)}}\underline{\phi}^{(n)^T}\underline{\phi}^{(n)}dt\right]\underline{T}^{(n)} - \left[\int_{\Omega_{el}^{(n)}}\underline{\phi}^{(n)^T}dt\right]\right\} = 0$$

- This leads to the following elemental matrices and vectors for each element *n*:

$$\underline{A}^{(n)} = \left[\int_{\Omega_{el}^{(n)}} \underline{\phi}^{(n)^T} \underline{\phi}_{,t}^{(n)} \, dt \right]$$

represents the elemental matrix associated with the time derivative

$$\underline{B}^{(n)} = \left[2 \int_{\Omega_{el}^{(n)}} \underline{\phi}^{(n)^T} \underline{\phi}^{(n)} \, dt \right]$$

represents the elemental matrix associated with the 2T terms in the o.d.e.

$$\underline{P}^{(n)} = \left[\int_{\Omega_{el}^{(n)}} \underline{\phi}^{(n)^T} \, dt \right]$$

represents the r.h.s. vector. In general the r.h.s. vector is produced by:

- p terms in the d.e. Recall $L(u) - p(x) = 0$
- boundary terms integrated over the boundary (natural b.c.'s).

- Thus

$$\sum_{n=1}^{\#el} \delta \underline{T}^{(n)^T} \left\{ \left(\underline{A}^{(n)} + \underline{B}^{(n)} \right) \underline{T}^{(n)} - \underline{P}^{(n)} \right\} = 0$$

- Summing over all the elements and taking into account the required functional continuity constraints leads to the global system of equations:

$$\delta \underline{T}^T \left\{ \left(\underline{A} + \underline{B} \right) \underline{T} - \underline{P} \right\} = 0$$

where $\underline{A}$ and $\underline{B}$ are global matrices while $\delta \underline{T}$, $\underline{T}$ and $\underline{P}$ are global vectors.

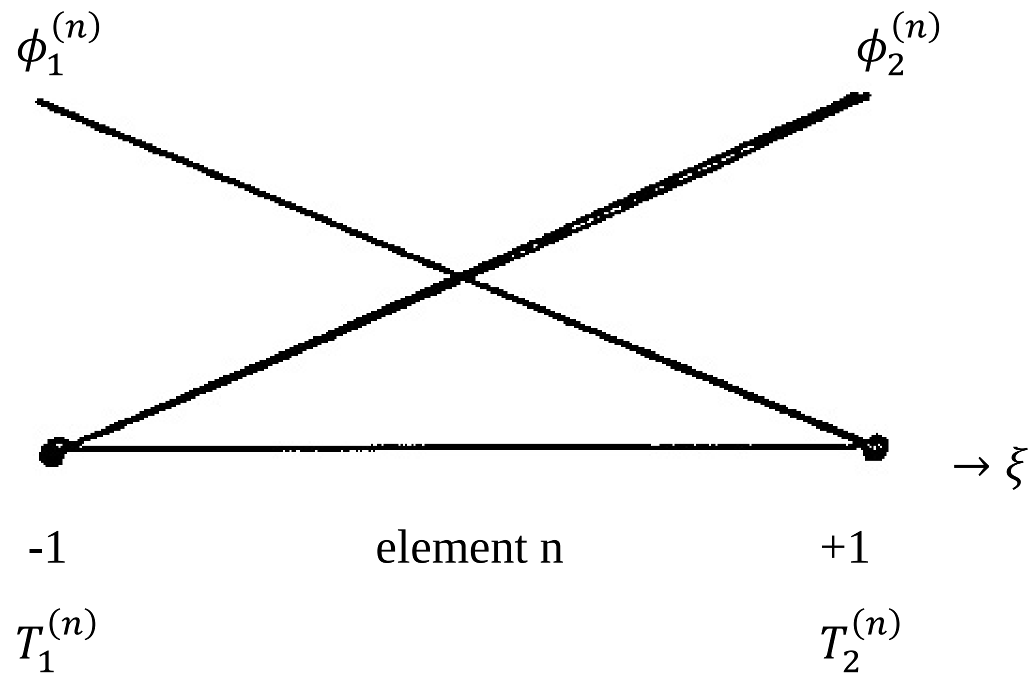
- The global matrices are formed with the local matrices.
- $\delta \underline{T}$ is a vector consisting of arbitrary coefficients δT_i. To ensure that the previous equation is true for any arbitrary set of coefficients δT_i (i.e. any arbitrary vector $\delta \underline{T}$) we <u>must have</u>:

$$\left(\underline{A} + \underline{B} \right) \underline{T} = \underline{P}$$

These equations and elemental matrices are valid for any order Lagrange interpolating function.

Let's develop the elemental matrices for Linear Lagrange elements

- Each element has 2 nodes, 2 interpolating functions and 2 unknown coefficients:



$$\underline{T}^{(n)} = \begin{bmatrix} T_1^{(n)} \\ T_2^{(n)} \end{bmatrix} \text{ and } \underline{\phi}^{(n)} = \begin{bmatrix} \phi_1^{(n)} \phi_2^{(n)} \end{bmatrix}$$

where

$\underline{T}^{(n)}$ equals a vector containing the unknown coefficients (i.e. the values of the function at the nodes in element *n*)

$\underline{\phi}^{(n)}$ equals a vector containing the interpolation functions for element *n*.

Specifically:

$$\phi_1^{(n)} = \frac{1}{2}(1 - \xi)$$

$$\phi_2^{(n)} = \frac{1}{2}(1 + \xi)$$

and therefore

$$\underline{\phi}^{(n)} = \left[\frac{1}{2}(1 - \xi)\,\frac{1}{2}(1 + \xi)\right]$$

- Recall transformation from global to local coordinates (unit element)

$$\xi = -1 + 2\left(t - t_j\right)/\left(t_{j+1} - t_j\right)$$

and vice versa

$$t = t_j + L_n(\xi + 1)/2$$

where L_n = length of element $n = t_{j+1} - t_j$

Let's develop $\underline{B}^{(n)}$

$$\underline{B}^{(n)} = 2 \int_{\Omega_e^{(n)}} \underline{\phi}^{(n)^T} \underline{\phi}^{(n)} dt$$

- Transforming into local coordinates we have:

$$dt = \frac{L_n}{2} d\xi$$

Integration limits become: $-1 \leq \xi \leq 1$

- Hence

$$\underline{B}^{(n)} = 2 \int_{-1}^{+1} \begin{bmatrix} \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1+\xi) \end{bmatrix} \begin{bmatrix} \frac{1}{2}(1-\xi) & \frac{1}{2}(1+\xi) \end{bmatrix} \frac{L_n}{2} d\xi$$

$$= L_n \int_{-1}^{+1} \begin{bmatrix} \frac{1}{4}(1-\xi)^2 & \frac{1}{4}(1-\xi^2) \\ \frac{1}{4}(1-\xi^2) & \frac{1}{4}(1+\xi)^2 \end{bmatrix} d\xi$$

$$= \frac{L_n}{4}\begin{bmatrix} \left.\frac{-(1-\xi)^3}{3}\right|_{-1}^{+1} & \left.\left(\xi - \frac{\xi^3}{\xi}\right)\right|_{-1}^{+1} \\ & \left.\frac{(1+\xi)^3}{3}\right|_{-1}^{+1} \end{bmatrix}$$

$$= L_n \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Since $L_n = \frac{1}{2}$ for all elements in this case

$$\underline{B}^{(n)} = \frac{1}{2}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Note that $\underline{B}^{(n)}$ is symmetrical.

Let's develop $\underline{A}^{(n)}$

$$\underline{A}^{(n)} = \int_{\Omega_e^{(n)}} \underline{\phi}^{(n)^T} \underline{\phi}_{,t}^{(n)} dt$$

- Transforming into local coordinates we have:

$$dt = \frac{L_n}{2} d\xi$$

$$\frac{d\phi^{(n)}}{dt} = \frac{d\phi^{(n)}}{d\xi}\frac{d\xi}{dt} = \frac{2}{L_n}\phi_{,\xi}^{(n)}$$

Limits of integration become $-1 \leq \xi \leq 1$

- Hence

$$\underline{A}^{(n)} = \int_{-1}^{+1} \underline{\phi}^{(n)^T} \underline{\phi}_{,\xi}^{(n)} \cdot \frac{2}{L_n} \cdot \frac{L_n}{2} \cdot d\xi$$

- We note that:

$$\underline{\phi}_{,\xi}^{(n)} = \frac{d}{d\xi}\left\{\left[\frac{1}{2}(1-\xi), \frac{1}{2}(1+\xi)\right]\right\} = \left[-\frac{1}{2}, \frac{1}{2}\right]$$

- Substituting:

$$\underline{A}^{(n)} = \int_{-1}^{+1} \begin{bmatrix} \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1+\xi) \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & +\frac{1}{2} \end{bmatrix} d\xi$$

$$= \int_{-1}^{+1} \begin{bmatrix} -\frac{1}{4}(1-\xi) & \frac{1}{4}(1-\xi) \\ -\frac{1}{4}(1+\xi) & \frac{1}{4}(1+\xi) \end{bmatrix} d\xi$$

$$= \frac{1}{4} \begin{bmatrix} \left(\frac{\xi^2}{2} - \xi\right)\Big|_{-1}^{+1} & \left(-\frac{\xi^2}{2} + \xi\right)\Big|_{-1}^{+1} \\ \left(-\frac{\xi^2}{2} - \xi\right)\Big|_{-1}^{+1} & \left(\frac{\xi^2}{2} + \xi\right)\Big|_{-1}^{+1} \end{bmatrix}$$

Thus:

$$\underline{A}^{(n)} = \frac{1}{2} \begin{bmatrix} -1 & +1 \\ -1 & +1 \end{bmatrix}$$

Hence the matrix $\underline{A}^{(n)}$ is identical for all elements regardless of their length L_n.

- For our particular case:

$$\underline{A}^{(n)} + \underline{B}^{(n)} = \frac{1}{2}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{5}{3} \end{bmatrix}$$

<u>Let's now develop $\underline{P}^{(n)}$</u>

$$\underline{P}^{(n)} = \left[\int_{\Omega_e^{(n)}} \underline{\phi}^{(n)^T} dt \right]$$

$$= \int_{-1}^{+1} \begin{bmatrix} \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1-\xi) \end{bmatrix} \frac{L_n}{2}\, d\xi$$

$$= \frac{L_n}{4}\begin{bmatrix} \left(\xi - \frac{\xi^2}{2}\right)\Big|_{-1}^{+1} \\ \left(\xi - \frac{\xi^2}{2}\right)\Big|_{-1}^{+1} \end{bmatrix}$$

$$= L_n \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

- For our particular case $L_n = \frac{1}{2}$ for both elements:

$$\underline{P}^{(n)} \frac{1}{2} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

- The elemental systems of equations are identical for both element 1 and 2:

$$\left(\underline{A}^{(n)} + \underline{B}^{(n)}\right)\underline{T}^{(n)} = \underline{P}^{(n)}$$

For element *n*:

$$\frac{1}{2} \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} T_1^{(n)} \\ T_2^{(n)} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

- Let's now form the global system of equations. Note that functional continuity at inter-element boundaries requires that:

elemental	global
$T_1^{(1)} =$	T_1
$T_2^{(1)} =$	T_2
$T_1^{(2)} =$	T_2
$T_2^{(2)} =$	T_3

- Contributions from each element are as follows to form the global system (we implement the summing of the elemental equations over all the elements placing each contribution in the proper global location):

$$\begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \\ -\frac{2}{3} & \left(\frac{5}{3} - \frac{1}{3}\right) & \frac{4}{3} \\ & -\frac{2}{3} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} + \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Now we set the essential b.c.'s by eliminating equation 1 and setting $T_1 = 1$.